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Path Integrals on Symmetric Spaces and Group Manifolds

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Invariant path integrals on symmetric and group spaces are defined in terms of a sum over the paths formed by broken geodesic segments. Their evaluation proceeds by using the mean value properties of functions over the geodesic and complex radius spheres. It is shown that on symmetric spaces the invariant path integral gives a kernel of the Schrödinger equation in terms of the spectral resolution of the zonal functions of the space. On compact group spaces the invariant path integral reduces to a sum over powers of Gaussian-type integrals which, for a free particle, yields the standard Van Vleck–Pauli propagator. Explicit calculations are performed for the case of $SU(2)$ and $U(N)$ group spaces.

1. INTRODUCTION

Feynman path integrals offer a deep insight into the relationship between classical and quantum phenomena. The extension of path integrals to non-Euclidean spaces provides a natural means of introducing nonlinear interactions and the interplay between the physics and the space geometry. For this reason there has been considerable interest in the use of Feynman integrals in curved spaces in many branches of physics. DeWitt (1957) showed how to extend the Euclidean Feynman path integral to a Riemannian space. The difficulties of expressing the path integral in non-Cartesian coordinates have been known since Edwards and Gulayev (1964) showed the role of the higher order terms in the Taylor expansion of the short-time action functional. The effect of this is that the Cartesian path integral is not invariant under ordinary coordinate transformations and depends also on the method of time slicing of the broken linear paths. This leads to alternative but equivalent definitions

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of non-Cartesian path integrals which all satisfy the same Schrödinger equation. DeWitt used Taylor's series expansion of a general action to show that the short-time propagator with the classical action satisfies the invariant Schrödinger equation provided a correction term proportional to the scalar curvature is added to the action. His form of short-time propagator uses the preexponential factor consisting of the square root of the classical Van Vleck determinant. This type of determinant, which arises in the theory of motion of a classical practice in the phase space, depends only on the classical action. DeWitt's short-time propagator in a curved space has the same form as the well-known Van Vleck–Pauli finite-time Green's function for quadratic systems in Euclidean space.

Schulman (1968) found that when the Riemannian space is the group space of $SU(2)$, which is isomorphic to a 3-dimensional sphere, the integral kernel of the invariant Schrödinger equation can be expressed as a sum over all multiple classical paths of the terms consisting of the Van Vleck–Pauli formula. This form of the finite-time propagator has been termed "exact" by Dowker (1971), who showed that the free particle propagator remains "exact" on all unitary group spaces. It is interesting to note that while in the Euclidean space the Van Vleck–Pauli formula remains valid for quadratic potentials, the exact form of the propagator in curved spaces seems to be found in the case of free particle action in locally compact group spaces only. It can be shown that the spaces of even higher symmetry than the general group spaces, for example, the two-dimensional sphere S_2, do not exhibit this property.

The objective of the present paper is to understand what particular feature of a group space enables us to write the exact expression for the free particle propagator. This problem has been addressed from various standpoints by many authors. Schulman used the well-known spectral Green's function expansion in a stationary series of eigenfunctions of the Laplace–Beltrami operator on $SU(2)$. This is just the theta function on a group space, which can be evaluated to yield the exact form. Dowker obtained the propagator by solving the Schrödinger equation in the group space of $SU(N)$. Other authors (for example, Bohm and Junker, 1987) explored the connection between the curved manifold and the Euclidean space of higher dimension which generates this manifold under the group transformation. These methods have been used to define path integrals on n-dimensional spheres and some group manifolds which can be immersed in Euclidean space of a higher dimension.

Our approach is to write down the path integral on a symmetric space in an invariant form. The summation is over the paths consisting of geodesic links confined entirely to the symmetric space. This is a generalization of the broken straight-line path used in Euclidean space. We develop a method of path integration in which the variables are the geodesic path segments.

This path integral is invariant under coordinate changes, so that we can dispense with the coordinate-dependent quantum correction term to the classical Lagrangian that is needed in the standard linearized approximation of the curvilinear path integral (e.g., Groshe, 1991). We show that the invariant path integral on a symmetric space is a limit of a random walk on the symmetric manifold and leads to a sum over the powers of Riemannian integrals of the zonal functions of the manifold. This is the equivalent of the standard formula for the spectral representation of the propagator in terms of eigenfunctions and energy eigenvalues on a compact manifold.

Next, we consider a compact group space, which is a special case of a compact symmetric manifold. The simplification of a path integral is due to the special form of the zonal functions on a compact group manifold: they are the characters of the finite-dimensional group representations and, according to Weyl (1973), can be expressed as linear combinations of exponentials in Cartan's radial coordinates. This reduces the functional integral to a multiple of independent Gaussian integrals which can be easily evaluated to yield the standard result for the closed form of a propagator on a compact group manifold (Dowker, 1971). We can show explicitly that the Markovian property of the exact propagators is satisfied on all compact group spaces, although the demonstration is given only for $SU(2)$ and $U(N)$.

The structure of the paper is as follows: In Section 2 we explain the central mathematical tool based on the work of Gelfand. This consists of a generalization of the notion of a mean of an integrable function $f(x)$ over a surface of a geodesic sphere $S_m(R)$ with the center at m and the radius R. In an N-dimensional Euclidean space the well-known Pizzetti formula (Courant and Hilbert, 1962) yields

$$M_m(R)f(x) = j_{N/2-1}(R\sqrt{\Delta_2})f(m)$$

where $M_m(R)$ is the averaging operator acting on the function $f(x)$ and $j_{N/2-1}(x)$ is the modified Bessel function with a symbolic argument involving the Laplace operator.

We show how to write a similar formula for $SU(2)$, namely

$$M_f(R)f(g) = \frac{1}{\sin \frac{1}{2}R} \frac{\sin\{\frac{1}{2}R\sqrt{1/4 - \Delta}\}}{\sqrt{1/4 - \Delta}} f(g)$$

For a mean value of central functions $\zeta(g)$ over a 2D sphere in $SU(2)$ of radius R and a center at radial distance θ from the origin, we have the alternative integral transform formula

$$M_\theta(R)\zeta(g) = \int_0^\infty a(\theta, R; u)\zeta(u)\, du$$

where the kernel $a(\theta, R; u)$ is given by (7).

In Section 3 we write down the invariant path integral on $SU(2)$ and use the mean value formula to evaluate the finite-time propagator. We derive several covariant expressions which show the relationship between the path integrals, short-time and finite-time propagators, and the Schrödinger equation. The functional integral allows us to write down a distribution for the one-dimensional radial process $\theta(t)$ on $SU(2)$:

$$\frac{e^{it/8}}{\sqrt{2\pi it}}\frac{\sin(\theta_1\theta_2/t)}{\sin\frac{1}{2}\theta_1\,\sin\frac{1}{2}\theta_2}\exp^{i\theta_1^2/2t}\exp^{i\theta_2^2/2t}$$

In Section 4 we outline the generalization of the path integrals to symmetric spaces. We also consider the spaces of rank higher than one, where instead of a geodesic sphere we are led to introduce the so-called sphere of complex radius. Unlike the rank-one spaces, here the complex radius is a function of the independent invariants of the group, the number of which equals to the rank of the space. We show that the free particle path integral will be exact if the zonal functions are essentially a sum of linearly independent exponentials of complex radius variables. This is always true for compact group spaces, where the zonal functions are just the characters of the finite-dimensional irreducible representations of the compact group. Section 5 provides an illustration in the form of the $U(N)$ group space.

The Appendix provides an overview of the definitions and the theorems for symmetric spaces used in the main text.

2. MEAN VALUE PROPERTIES

Godement (1952) was first to show that the zonal functions $\zeta_\nu(y)$ on any symmetric space $M = G/H$ satisfy a functional equation of the form

$$\zeta_\nu(x)\cdot\zeta_\nu(r) = \int_H \zeta_\nu(x, h, r)\, dh \tag{1}$$

where x and $r \in G$, $h \in H$, and dh is the invariant volume element of H.

Berezin and Gelfand (1962) realized that the integral on the right-hand side of the above identity can be interpreted geometrically as a mean value of $\zeta_\nu(y)$ over a surface of a sphere $S(x, r)$ with complex radius r and center at x. Thus we can write

$$\zeta_\nu(x)\cdot\zeta_\nu(r) = \frac{1}{\mu(S)}\int_{y\in S(x,r)} \zeta_\nu(y)\,\mu(dy) \equiv M_x(r)\zeta_\nu(y) \tag{2}$$

where $\mu(S)$ is the surface area of the sphere $S(x,r)$.

It can be shown that if the space M is a group space of a compact Lie group G, then, up to a constant multiplier, the zonal spherical functions are the characters $\phi^{\nu}(g)$ of the ν-dimensional irreducible representation of G.

We now consider the group space of $SU(2)$, the group of complex unimodular matrices of second order. Every unimodular matrix may be brought by a similarity transformation to diagonal form with the diagonal entries $\epsilon_1 = \epsilon^{i\theta}$ and $\epsilon_2 = e^{-i\theta}$. The complex distance is a single parameter θ which lies in a closed interval between 0 and 2π and represents the geodesic distance from the origin.

The invariant integral measure on $SU(2)$ can be written as a product of an angular part corresponding to the similarity transformations and a polar part, which is the measure on diagonal matrices given by the formula $du[\epsilon] = 4\sin^2(\frac{1}{2}\theta)\, d\theta$.

The zonal functions $\zeta_\nu(g)$ are the characters of $SU(2)$ divided by the dimension of the representation

$$\zeta_\nu(\epsilon) = \frac{\sin\{(2\nu + 1)\theta/2\}}{(2\nu + 1)\sin\theta/2} \tag{3}$$

where $\nu = 0, 1/2, 1, 3/2$, etc.

The functional relation (2) between the zonal functions takes on a simple form,

$$M_\theta(r)\zeta_\nu(g) = \zeta_\nu(r)\zeta_\nu(\theta) \tag{4}$$

Here the mean is over a sphere of radius r with a center at the distance θ from the origin.

The Laplace–Beltrami operator Δ on $SU(2)$ is invariant under right and left translations. It can be expressed in the form (Berezin, 1962)

$$\Delta = \Delta_\theta + \frac{1}{\theta^2}\Delta_\omega$$

where

$$\Delta_\theta = \frac{1}{\sin\frac{1}{2}\theta}\frac{\partial^2}{\partial\theta^2}\sin\frac{\theta}{2} - \frac{1}{4}$$

is the radial part of the Laplace operator and Δ_ω is the angular part, which does not involve any derivatives with respect to the polar coordinate θ. The constant factor of 1/4 is equal to 1/6 of the scalar curvature R.

The zonal functions are eigenfunctions of the Laplace–Beltrami operator

$$\Delta\zeta_\nu(g) = \Delta_\theta\zeta_\nu(\theta) = \Lambda(\nu)\zeta_\nu(g)$$

with the eigenvalue

$$\Lambda(\nu) = \frac{1}{4} - \left(\frac{(2\nu + 1)}{2}\right)^2 = \nu(\nu + 1)$$

This allows us to rewrite (4) in the symbolic expression

$$M_\theta(r)\zeta_\nu(\theta) = \frac{1}{\sin \frac{1}{2}R} \frac{\sin\{\frac{1}{2}r\sqrt{1/4 - \Delta_\theta}\}}{\sqrt{1/4 - \Delta_\theta}} \zeta_\nu(\theta)$$

Now, any integrable function on the group space of $SU(2)$ may be approximated by a linear combination of zonal functions and its translates. Therefore, the validity of the above formula extends to all integrable functions $f(g)$ if we replace the radial part of the Laplace operator by Δ:

$$M_\theta(r)f(g) = \frac{1}{\sin \frac{1}{2}r} \frac{\sin\{\frac{1}{2}r\sqrt{1/4 - \Delta}\}}{\sqrt{1/4 - \Delta}} f(g) \tag{5}$$

Recalling that $j_{1/2}(x) = (\sin x)/x$, we see that the above formula resembles the Pizzetti formula for E_3.

If we apply formula (5) to spherical functions $\psi_\nu(g)$, we obtain a stronger version of (4), showing that also the spherical harmonics are eigenfunctions of the averaging operator

$$M_{g_0}(r)\psi_\nu(g) = \zeta_\nu(r)\psi_\nu(g_0) \tag{6}$$

We now give an alternative integral expression for the mean on a sphere valid for central functions only. Start with the simple integral identity

$$\zeta_\nu(\theta_1)\zeta_\nu(\theta_2) = \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} \frac{\sin \frac{1}{2}u}{4 \sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2} \zeta_\nu(u)\, du$$

which follows by substitution from (3). We see that according to (4), the left-hand side of the above identity is the mean of $\zeta_\nu(u)$ over a sphere of radius θ_2 with center at the distance θ_1. As the integral operator on the right-hand side acts linearly on $\zeta_\nu(u)$ and as every zonal function can be represented as a linear combination of characters, the right-hand side must be valid for any central function $\zeta(g)$.

Thus we have

$$M_{\theta_1}(\theta_2)\zeta(g) = \int_0^\infty a(\theta_1, \theta_2; u)\zeta(u)\, du$$

where the kernel of the integral operator is

$$a(\theta_1, \theta_2; u) = \frac{\sin \frac{1}{2}u}{4 \sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2} \quad \text{for} \quad \theta_1 - \theta_2 \leq u \leq \theta_1 + \theta_2$$
$$a(\theta_1, \theta_2; u) = 0 \quad \text{otherwise} \tag{7}$$

3. PATH INTEGRALS ON *SU*(2) SPACE

3.1. Invariant Definition of Path Integral

The propagator of the Schrödinger equation can be represented in the form of the Feynman path integral $\int_x^y \exp\{iS(q_t)\}\, d[q_t]$, where the integral over the measure $d[q_t]$ signifies the "sum" of all paths q_t between the fixed endpoints x and y. This can be rewritten as a limit of an infinite Markovian chain

$$\lim_{n\to\infty} \int_{q_1\cdots q_{n-1}} k(x, q_1; \delta t)\, d[q_1]\, k(q_1, q_2; \delta t) \times \cdots k(x, q_{n-1}; \delta t)\, d[q_{n-1}]\, k(q_1, y; \delta t) \tag{8}$$

consisting of the short-time propagators $k(q, q'; \delta t) = (2\pi i \delta t)^{-3/2} \exp\{iS(q, q'; \delta t)\}$. The exponential term $S(q, q'; \delta t)$ is an approximation to the classical action between q and q' over the short time interval $\delta t = t/n$. For a free particle the classical action $S = \frac{1}{2} d^2(q, q')/\delta t$ depends only on the square of the geodesic distance between the endpoints.

The form of path integral (8) lends itself to the following interpretation in terms of a random walk, adapted from Roberts and Ursell (1960). We start with an initial space distribution in the form of a delta function $\psi^{(0)}(x) = \delta(x, x_0)$ centered at a point x_0 and define a random walk starting at point x_0 and subject to the following rules:

1. The process changes the initial distribution in discrete finite time steps of duration δt.

2. The value at x of a new distribution after k steps is the mean of the previous distribution after $k - 1$ steps calculated over the surface of a sphere with a radius θ_k centered at x.

Thus the distribution after n steps will be

$$\psi^{(n)}(x) = \prod_{k=1}^{n} M_x(\theta_k)\delta(x, x_0)$$

Expressing the delta function as a sum of pairs of orthonormal spherical functions $\psi_\nu(x)$ and using the identity (4), we obtain

$$\psi^{(n)}(x) = \sum_\nu \prod_{k=1}^{n} \zeta_\nu(\theta_k)\psi_\nu(x)\psi_\nu(x_0) \tag{9}$$

This represents the sum of all those broken geodesic paths from x_0 to x which are defined by $\{\theta_k\}$, a sequence of the lengths of their geodesic segments.

If we chose the segment lengths to be small independent random variables and ask only that their second variance satisfies the condition

$$E(\theta_k^2) = 3\delta t + O(\delta t^2) \tag{10}$$

then we obtain, in the limit of a large number of steps, a random Brownian motion on $SU(2)$. To see this, expand (3) in the small arguments θ_k

$$\zeta_\nu(\theta_k) \approx 1 + \tfrac{1}{3}\lambda(\nu)\theta_k^2$$

and substitute into (9). The result is

$$\psi^{(\infty)}(x) \equiv \psi(x, t) = \sum_\nu e^{it\lambda(\nu)/2}\, \psi_\nu(x)\psi_\nu(x_0)$$

which is the kernel of the Schrödinger equation.

On the other hand, in order to obtain a path integral form of (9), we ask that the distribution density of the geodesic arcs θ_k be equal to

$$k(\theta_k, t) = (2\pi it)^{-3/2} \exp\{\tfrac{1}{2}i\theta_k^2 - \tfrac{1}{4}it\}$$

we obtain from (9)

$$\psi^{(n)}(x) = \sum_\nu [\kappa_\nu(t)]^n \psi_\nu(x)\psi_\nu(x_0)$$

with $\kappa_\nu(t)$ equal to

$$\int_0^\infty k(\theta, t)\zeta_\nu(\theta) \sin^2\theta/2 \, d\theta$$

This is a simple Gaussian integral which gives

$$\kappa_\nu(t) = \frac{\sin(t\sqrt{\nu(\nu + 1)})}{t\sqrt{\nu(\nu + 1)}} \exp\left\{i\left[\frac{t}{4} + \frac{1}{2}t\nu(\nu + 1)\right]\right\}$$

If we increase the number of steps n to infinity, $t \to t/n$ and $[\kappa_\nu(t/n)]^n \to \exp[\tfrac{1}{2}it\Lambda(\nu)]$, so that the limit of the random walk with the distribution $k(\theta_k, t)$ is again a Brownian motion.

3.2. Integral Kernel

The differential operator $\exp(\tfrac{1}{2}it\Delta)$ acting on the function $\Psi(x)$ gives a time-dependent solution of Schrödinger equation

$$-\frac{1}{2}\,\Delta\Psi(x, t) = i\,\frac{\partial}{\partial t}\,\Psi(x, t)$$

In this section we will show how to express this operator in an integral form with a kernel distribution $K(x, y; t)$ such that

$$\exp\left\{\frac{it}{2}\,\Delta\right\} f(x) = \int K(x, y; t) f(y)\, d[y]$$

where $d[y]$ is the invariant measure on the group manifold and the integration extends over the whole of $SU(2)$.

Using formally the Gaussian-type integral identity

$$\int_0^\infty x \sin(mx) \exp[-ia^2x^2\}\, dx = \frac{m\sqrt{i\pi}}{4a^3} \exp\left\{-i\,\frac{m^2}{4a^2}\right\}$$

we proceed as follows:

$$\exp\left\{\frac{it}{2}\,\Delta\right\} = \exp\left\{\frac{it}{8}\right\} \exp\left\{-\frac{it}{2}\left(\frac{1}{4} - \Delta\right)\right\}$$

$$= \exp\left\{\frac{it}{8}\right\} \int_0^\infty dr\, \frac{r \sin(\frac{1}{2}r\sqrt{1/4 - \Delta})}{\sqrt{1/4 - \Delta}} \exp\left\{\frac{it^2}{2t}\right\} \frac{2\pi}{(2\pi it)^{3/2}}$$

Rearranging the terms, we obtain

$$\exp\left\{\frac{it}{2}\,\Delta\right\} f(x)$$

$$= \exp\left\{\frac{it}{8}\right\} \int_0^\infty d[r]\, \frac{\frac{1}{2}r}{\sin\frac{1}{2}r}\, \frac{1}{(2\pi it)^{3/2}}$$

$$\times \exp\left\{\frac{ir^2}{2t}\right\}\left(\frac{\sin(\frac{1}{2}r\sqrt{1/4 - \Delta})}{\sqrt{1/4 - \Delta}\,\sin\frac{1}{2}r} f(x)\right)$$

where $d[r] = 4\pi(\sin\frac{1}{2}r)^2\, dr$.

But the expression

$$\left(\frac{\sin(\frac{1}{2}r\sqrt{1/4 - \Delta})}{\sqrt{1/4 - \Delta}\,\sin\frac{1}{2}r} f(x)\right)$$

represents $M_x(r)f(g)$, a mean of $f(x)$ over the surface of a sphere with center at x and radius r. In other words, we have an invariant expression for an integral over the volume of the entire group, covering it an infinite number of times over. This represents a summation over the winding number of equivalent geodesics between two fixed points.

With this in mind we can write

$$\exp\left\{\frac{it}{2}\Delta\right\} f(x) = \int K(x, y; t) f(y)\, d[y]$$

where

$$K(x, y; t) = \sum_{n=0}^{\infty} \frac{1}{(2\pi i t)^{3/2}} \frac{\frac{1}{2}d(x, y) + \pi n}{\sin[\frac{1}{2}d(x, y) + \pi n]} \exp\left\{\frac{i(d(x, y) + \pi n)^2}{2t}\right\} \exp\left\{\frac{it}{8}\right\}$$

where $d(x, y)$ is the shortest geodesic distance between x and y.

This form of kernel coincides with the semiclassical Van Vleck–Pauli form, which can be written as a sum of

$$K_{\text{clas}}(x, y; t) = D(x, y; t)^{1/2} e^{iS(x,y;t)}$$

where $S(x, y; t)$ is the classical action and the preexponential term is the square root of the Van Vleck determinant $D(x, y; t) = g(x)^{-1/2} \det[\partial_x \partial_y(S)]\, g(y)^{-1/2}$. We shall call this form of a propagator exact.

Next, we show how we can explicitly check that the exact propagator has the Markovian property

$$\int K(x_0, x_1; t_1)\, d[x_1]\, K(x_1, x_2, t_2) = K(x_0, x_2; t_1 + t_2)$$

Here we have a three-point broken geodesic path with the starting point x_0 placed at the origin. The middle point x_1 and the fixed endpoint x_2 are at distances θ_1 and θ from the origin, respectively. These three points form a broken geodesic path consisting of two geodesic segments x_0x_1 and x_1x_2 whose lengths are θ_1 and θ_2, respectively. The integral over the group space is then to be taken as an integral over a volume of a sphere of radius θ_2 with the center at the endpoint x_2. The left-hand side can then be written as

$$\int_0^{\infty} [M_{x2}(\theta_2)K(\theta_1; t_1)]K(\theta_2; t_2) \sin^2 \tfrac{1}{2}\theta_2\, d\theta_2$$

First we calculate the mean over this sphere, taking a finite radius θ_2. Using formula (7) for the mean of the central function $K(\theta_1; t)$, we can write

$$M_{x2}(\theta_2)K(\theta_1; t_1) = \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} a(\theta_1, \theta_2, u)K(u; t_1)\, du$$

A simple calculation shows this integral to be

$$\frac{e^{it/8}}{\sqrt{2\pi i t_1}} \frac{\sin(\theta_1\theta_2/t_1)}{\sin\frac{1}{2}\theta_1 \, \sin\frac{1}{2}\theta_2} e^{i\theta_1^2/2t_1} e^{i\theta_2^2/2t_1}$$

This is the radial distribution on $SU(2)$, which is the equivalent of the Bessel process in 3D Euclidean space. Alternatively, the above expression may be written in the form

$$K(\theta_1;\, t_1)\zeta_\alpha(\tfrac{1}{2}\theta_2)e^{i\theta_2^2/2t_1} \qquad \text{where} \quad \alpha = 2\theta_1/t_1 \tag{11}$$

The integration over θ_2 leads to a simple Gaussian integral from which one shows easily that the Markovian property is satisfied.

4. PATH INTEGRALS ON SYMMETRIC SPACE

The previous definition of path integrals on $SU(2)$ may be extended to a symmetric space without significant changes. All that is needed is to replace the geodesic sphere with a complex sphere, in case the symmetric space has a rank larger than one.

The random walk distribution after k jumps at the time $t_k = k\,\delta t$ will be

$$\psi^{(k)}(x) = \prod_{j=1}^{k} \int \rho_{\delta t}(\theta_k)\; \mu(d\theta_k)\; M_x(\theta_k)\; \delta(x, x_0)$$

This formula represents a sum over certain types of continuous paths between x_0 and x. These paths consist of k geodesic segments of successive complex lengths $s_1, \ldots, s_k$ distributed in accordance with the distribution $\rho_{\delta t}(\theta)$.

Using formula (2) and the expression for the delta function as a sum of spherical functions, we get

$$\begin{aligned}
\psi^{(k)}(x) &= \prod_{j=1}^{k} \int \rho_{\delta t}(\theta_k)\; \mu(d\theta_k)\; M_x(\theta_k) \sum_{\nu} \phi_\nu(x)\phi_\nu(x_0) \\
&= \sum_{\nu} \phi_\nu(x)\phi_\nu(x_0) \prod_{j=1}^{k} \int \rho_{\delta t}(\theta_k)\; \mu(d\theta_k)\; \phi_\nu(\theta_k) \\
&= \sum_{\nu} (\kappa(\delta t,\, \nu))^k \phi_\nu(x)\phi_\nu(x_0)
\end{aligned}$$

where

$$\kappa(\delta t,\, \nu) = \int \rho_{\delta t}(\theta)\zeta_\nu(\theta)\; \mu(d\theta) \tag{12}$$

The path integral is defined by the limit $k \to \infty$, whereby $\delta t = t/k$ becomes an infinitesimal time. All that is needed is to prove that this expression is a kernel of the Schrödinger equation. In order to show this, introduce the

Riemannian (normal) coordinates y^j based at x. In these coordinates the curves $y^j = s\xi^j$, determined by the unit vector ξ^j at x, are geodesics with the parameter s representing the geodesic distance from the point x. Taking the distribution to be the infinitesimal Feynman propagator, we obtain

$$\rho_{\delta t}(\theta) = \frac{1}{(2\pi i\delta t)^{N/2}} e^{is^2/2\delta t + iR\delta t/6}$$

where R is the scalar Riemannian curvature.

Using the Taylor expansion at x to approximate the zonal function and the elementary volume element, we obtain to the second order in y^j

$$\zeta_\nu(\theta) \approx \zeta_\nu(0) + \tfrac{1}{2}\zeta_\nu(0)_{,nm}y^n y^m$$

$$\mu(d\theta) = \sqrt{g(\theta)}\, d\theta_1 \cdots d\theta_N \approx \sqrt{g(0)}(1 + \tfrac{1}{6}R_{nm}y^n y^m)\, dy^1 \cdots dy^N$$

Here R_{nm} is the Ricci tensor at x and the subscripts after the comma denote partial differentiation with respect to the Riemannian coordinates. The quantities thus obtained are in fact the components of invariant tensors. It is now apparent that if we substitute these expansions into (12) we obtain to the first order in δt

$$\kappa(\delta t, \nu) = 1 + \tfrac{1}{2}i\, \delta t\, g(0)^{nm}\zeta_\nu(0)_{,nm}$$

As is well known, the second partial derivatives with respect to the Riemannian coordinates are equal to the second-order covariant derivatives in an arbitrary coordinate system

$$\kappa(\delta t, \nu) = 1 + \tfrac{1}{2}i\, \delta t\, \Delta\zeta_\nu(0) = 1 + \tfrac{1}{2}i\Lambda(\nu)\, \delta t$$

where $\Lambda(\nu)$ is the eigenvalue of the Laplace–Beltrami operator corresponding to the eigenfunction $\zeta_\nu(x)$.

Therefore the path integral

$$\lim_{k\to\infty} \psi^{(k)}(x) = \sum_\nu (\kappa(t/k, \nu))^k \phi_\nu(x)\phi_\nu(x_0)$$

$$= \sum_\nu e^{it\Lambda(\nu)/2}\phi_\nu(x)\phi_\nu(x_0)$$

is the kernel of the Schrödinger equation.

For the path integral to be exact a more stringent condition is required, namely

$$\kappa(t, \rho) = e^{i\Lambda(\rho)t/2} \tag{13}$$

for any finite t. This condition is not satisfied for an arbitrary symmetric space, but is always true when the symmetric space happens to be a compact group space. This is because the zonal function $\zeta_\nu(\theta)$ on a group space is the character of the group and is essentially a sum of exponentials of the form $\exp(i\mathbf{m}\cdot\theta)$, where $\mathbf{m}$ is a vector consisting of integers or half-integers. Therefore the evaluation of the coefficients $\kappa(t, \nu)$ reduces to the calculation of Gaussian integrals

$$\kappa(t, \nu) = \int \rho_t(\theta)\zeta_\nu(\theta)\ \mu(d\theta)$$

which have the form of (13).

5. EXAMPLE OF $U(N)$

Consider an N^2-dimensional group space of $U(N)$. The rank is N. We employ the standard multi-index notation whereby $\mathbf{x} \equiv (x_1, \ldots, x_N)$ and $\mathbf{x}^\alpha \equiv \Pi_{k=1}^N x_k^{\alpha_k}$. Cartan decomposition allows the introduction of N polar coordinates θ which correspond to H, the maximum Abelian subgroup of diagonal matrices with eigenvalues $\epsilon = \exp\{i\theta\}$. Geometrically they represent an N-dimensional subspace, the so-called invariant torus. The points h lying on this torus can be transformed into each other by interchanging the position of the eigenvalues, the so-called Weyl transformation. The inner automorphism of the group $h \to g^{-1}\cdot h\cdot g$, where g is an arbitrary element of the group, constitutes a rotation around the origin, so that the points with the same set (up to their permutation) of polar coordinates θ lie on the same $N(N - 1)$-dimensional complex sphere. This set of coordinates is called the complex radius. The geodesic radius s given by $s^2 = c\theta^2$ (Biederharn, 1963), where the scaling constant c may be taken as unity.

According to Weyl (1973), characters of the representation of a unitary group are given by the maximum vector $\boldsymbol{\rho} = (\rho_1, \rho_2 \cdots \rho_n)$ consisting of the ordered integers $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \geq 0$. With the notation

$$\Pi(\mathbf{x}) \equiv \prod_{i<j} (x_j - x_i) \qquad \text{and} \qquad \Pi_{\boldsymbol{\rho}}(\mathbf{x}) \equiv \sum_{\sigma(\boldsymbol{\rho})} \sigma(\boldsymbol{\rho}) x_1^{\rho_1} x_2^{\rho_2} \cdots x_N^{\rho_N}$$

where $\sigma(\boldsymbol{\rho})$ denotes the antisymmetric summation over all permutations of the vector components of $\boldsymbol{\rho}$, we can express the characters of $U(N)$ in the form

$$\phi_{\boldsymbol{\rho}}(\theta) = \frac{\Pi_{\boldsymbol{\rho}}(e^{i\theta})}{\Pi(e^{i\theta})}$$

The dimension of the representation is

$$d_{\boldsymbol{\rho}} = \frac{\Pi(\boldsymbol{\rho})}{(n - 1)!(n - 2)! \cdots 1}$$

The zonal functions are the characters divided by the dimension of the representation $\zeta_\rho(\theta) = d_\rho^{-1}\phi_\rho(\theta)$. They are the eigenfunctions of the Laplace–Beltrami operator corresponding to the eigenvalues $\Lambda(\rho) = \Sigma_j^n \rho_j^2 - R/6$, with R being the scalar curvature of $U(N)$, equal to $\frac{1}{2}N(N^2 - 1)$ (e.g., Marinov and Terentyev, 1979).

The invariant measure on the maximal torus is

$$\mu(d\theta) = \frac{1}{A}\,\Pi(e^{i\theta})^2 d\theta_1 \cdots d\theta_N$$

with the normalization constant $A = (\pi)^{-p} n! \cdot (n - 1)! \cdots 1$ chosen to give unity volume over the $U(N)$ manifold. Here $2p = (N^2 - N)$ is the dimension of the angular component of the group space.

We need the form of the exact propagator $\rho_t(\theta)$ for $SU(N)$. This can be found by generalizing the radial distribution (11) to

$$K(\theta_1; t)\zeta_\alpha(\theta_2)e^{i\Sigma\theta_2^2/2t} \qquad \text{where} \quad \alpha \equiv \theta_1/t$$

In the limit $\theta_1 \to 0$ we obtain the radial process starting from the origin, which is the sought expression for $\rho_t(\theta)$. Carrying out the limiting process, we obtain

$$\rho_t(\theta) = \frac{\Pi(\theta)}{\pi(e^{i\theta})}\,\frac{e^{i\Sigma\theta^2/2t - itR/12}}{(2\pi t i)^{N^2/2}}$$

where we have used

$$K(0; t) = \frac{1}{(2\pi i t)^{N^2/2}}\,e^{iRt/12}$$

and the identity

$$\lim_{\alpha\to 0} \phi_\alpha(\theta) = \frac{\Pi(\theta)}{\Pi(e^{i\theta})}$$

The finite kernel has to be summed over all winding geodesics. This form of kernel has been obtained by Dowker (1971) by solving the Schrödinger equation for $SU(N)$.

We are now ready to calculate the "Fourier" coefficient $\kappa(t, \nu)$ given by the formula (12):

$$\kappa(t, \nu) = \frac{\pi^p}{N}\int \prod_{j<k} \frac{\theta_j - \theta_k}{\rho_j - \rho_k} \sum_{\sigma(\rho)} (\sigma(\rho)e^{i\rho_1\theta_1}e^{i\rho_2\theta_2} \cdots e^{i\rho_N\theta_N})\,\frac{e^{i\Sigma\theta^2/2t - iRt/12}}{(2\pi t i)^{N^2/2}}\, d\theta_1 \cdots d\theta_N$$

To evaluate this integral, we use the formal identity

$$\prod_{j<k} (\theta_j - \theta_k)^{\Sigma\sigma\, \exp(i\rho\cdot\theta)} = \frac{1}{i^p} \prod_{j<k} \left(\frac{\partial}{\partial\rho_j} - \frac{\partial}{\partial\rho_k}\right)^{\Sigma\, \exp(i\rho\cdot\theta)}$$

This leaves us with the multiple Gaussian integral

$$\frac{\pi^p}{i^p N(2\pi i t)^{N^2/2}\Pi(\rho_j - \rho_k)} \prod_{j<k} \left(\frac{\partial}{\partial\rho_j} - \frac{\partial}{\partial\rho_k}\right) \int \Sigma\, \exp(i\rho\cdot\theta)\, \exp[i(\Sigma\, \theta^2)/2t - iRt/12] \prod_j d\theta_j \cdot$$

The right-hand side factorizes into a product of Gaussian integrals. The expression

$$\frac{1}{(it)^p N\Pi(\rho_j - \rho_k)} \prod_{j<k} \left(\frac{\partial}{\partial\rho_j} - \frac{\partial}{\partial\rho_k}\right)$$

yields

$$\kappa(t, \rho) = e^{it\Lambda(\rho)/2}$$

This agrees with (13), so the propagator is exact.

APPENDIX. SYMMETRIC SPACES

We briefly state some facts about homogeneous and symmetric spaces. For a thorough discussion of these topics the reader is referred to more authoritative sources such as Helgason (1978) and Kobyashi and Nomizu (1962).

Let us consider G to be a transformation group (also called the group of rigid motions) of a topological space M. The elements of the group $g \in G$ act on the points of the space x: $x \to g\cdot x$. We shall be concerned only with transitive transformation groups which guarantee that for two arbitrary points of M there is always a transformation which sends one into another. Then M becomes a homogeneous space.

Let H_x be a stationary subgroup of x, that is, a subgroup of G which leaves the point x unchanged. Consider another point y. Then, because of the homogeneity of M, there is a transformation g such that $y = g\cdot x$. In terms of H_x, the stationary group H_y at y is given by the conjugate subgroup $g\cdot H_x \cdot g^{-1}$. Thus we need to know only the stationary group H_0 at one point, call it x_0 (the origin of M), to generate a stationary subgroup of any point of M. Given a rigid motion g, we can form gH_0, the left coset of g with respect to H_0. Then all the rigid motions belonging to this coset transform the point x_0 into the same point $g\cdot x_0$. This shows a one-to-one correspondence between the points of the homogeneous space M and the classes of left cosets with respect to H and allows an alternative view of M as the factor space G/H.

In this picture the rigid motion $g' \in G$ acting on a coset $g \cdot H$ produces another coset $g' \cdot g \cdot H$.

Let $T^{\nu}(g)$ be a ν-dimensional irreducible representation of G in a Hilbert space $\mathcal{H}$. If there is a vector $a_0 \in \mathcal{H}$ such that for any $h \in H_0$ the action of $T^{\nu}(h)$ leaves the vector a_0 stationary and the restriction of $T(g)$ to H is unitary, then $T^{\nu}(g)$ is called the representation of class 1 relative to the stationary subgroup H_0. Using such a representation, we can define a group function $\phi_{b\nu}(g)$ by forming the scalar product on $\mathcal{H}$ of an invariant vector a_0 with a vector b, where both a_0, $b \in \mathcal{H}$:

$$a_{b\nu}(g) \equiv \langle b, T^{\nu}(g) \cdot a_0 \rangle$$

This is a function on G/H, the so-called spherical function of dimension ν.

Specifically, if we choose $b = a_0$, then we obtain the so-called spherical zonal function ζ_{ν}. The zonal function is also defined on G/H, but in addition is a type of function which is constant on the two-sided cosets with respect to the stationary subgroup H_0. For any element $g \in G$ a two-sided coset of H_0 is formed by a set of all elements $h \cdot g \cdot h'$, where h and h' belong to H_0. The two-sided coset forms a submanifold in G/H which is called a sphere with center x_0 (represented by coset $e \cdot H_0$) and going through the point represented by the coset $g \cdot H_0$. To any two points on the same sphere we can assign an invariant function on the sphere. The number r of independent invariant functions is called the rank of the homogeneous space and their totality represents the complex radius of the sphere. Similarly, complex distance between two points is defined as the complex radius of the sphere formed by one point being at the center and the other lying on the sphere. Consequently, the zonal spherical functions are constant on the spheres with center determined by the stationary subgroup H_0 and depend on r arguments.

We now turn to symmetric spaces. These are homogeneous spaces described above with a symmetry restriction on the stationary subgroup H_0. For space M to be symmetric, the stationary subgroup H_0 has to be capable of being defined purely in terms of an involutory automorphism * of the rigid motion group G in such a way that H_0 consists only of those elements of G which are constant under the action of *.

A significant property of a symmetric space M is that it can be embedded in the group of its rigid motions in the form of a connected component G which includes the unit element. The connected component of G consists of all elements which satisfy the identity $g^* = g^{-1}$. If we use the same symbol M for this component, then the action of the group of rigid motions G on the symmetric space M is given by $g^* \cdot M \cdot g^{-1}$.

A simple example of a symmetric space is a group space. We consider group spaces of semisimple Lie groups, with particular emphasis on $U(N)$ and $SU(N)$. If we identify M with $U(N)$, then the group of rigid motions is

$G = U(N) \times U(N)$, the group of a combinations of left and right group multiplications on $U(N)$ with group elements of $U(N)$. For the action of G we can write $G \cdot M = (g, g') \cdot M = g^{-1}Mg'$, where g and g' belong to $U(N)$. The stationary subgroup of unity is $H_0 = (g, g)$. This is an invariant subgroup, so the factor space G/H is a group isomorphic to $U(N)$. The sphere through an arbitrary point g_1 of $U(N)$ is the right coset of $U(N)$ with respect to H_0 which consists of all the elements of the form $g^{-1} \cdot g_1 \cdot g$.

As is known, every unitary matrix g_1 can be brought to a diagonal form $\epsilon \equiv [\epsilon_j]$ by the above similarity transformations using unitary matrices g. Hence the element g_1 lies on the same radius sphere as the diagonal matrix element ϵ. In fact, the ordering of eigenvalues in any particular matrix can be changed by the so-called Weyl transformation S within the group, so all the diagonal matrices with the same set of eigenvalues lie on the same sphere. The N eigenvalues of the diagonal matrix form the complex radius of the sphere. The rank of the space is N. Similarly, for $SU(N)$, the additional condition on the value of the determinant reduces the rank of the space to $N - 1$.

We refer to Cartan decomposition of semisimple groups. The Cartan generators are diagonal matrices which correspond to the maximum Abelian subgroup, the so-called maximum torus T_r. The dimension of T_r coincides with the rank of the group.

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